

Multi-Probe Array Design for Partially-Coherent Phase Retrieval in Near-Field Measurements

Jonas Kornprobst, Alexander Paulus, Josef Knapp, and Thomas F. Eibert
Department of Electrical Engineering
School of Computation, Information, and Technology
Technical University of Munich, 80290 Munich, Germany
e-mail: {j.kornprobst;hft}@tum.de

Abstract—Phase retrieval, in particular for the operators arising from near-field measurements, is a non-convex task suffering from a severe lack of reliability due to local minima and false solutions. Approaches to tackle this issue, in turn, suffer from strict and often unrealistic oversampling requirements and possibly unfeasible computational complexities—in particular when larger scenarios are considered. In this paper, we analyze the sampling requirements for convex phase retrieval based on partially coherent observations which are captured with multi-probe arrays. We discuss requirements for the orientation and positioning of the probe antennas in the probe arrays. Following these conditions, a recently introduced linearized method is able to reconstruct a unique global solution reliably. The theoretical deliberations are corroborated with simulated near-field data.

Index Terms—Magnitude-only antenna measurements, phaseless, antenna arrays, field transformations, source reconstruction.

I. INTRODUCTION

Field measurements are a generally important concept in radio-frequency (RF) engineering, be it for antennas, scattering, imaging, or other areas. In situations where measurements with full phase information are not feasible, e.g., due to difficult mechanical access to the feed of an antenna or phase instability at high frequencies, we are restricted to measure the magnitudes of the electromagnetic fields [1]–[8]. Typically, phase retrieval techniques are then employed to process the measurement data.

Our area of interest are magnitude-only near-field (NF) antenna measurements [9]–[24] and imaging [25], [26], but insights should be transferable to other areas, too. Since the properties of the forward operator are not as favorable as for Gaussian random matrices, which are often employed in mathematical studies of phase retrieval algorithms [27], [28], numerous problems occur in the solution process. Even with extreme oversampling and known uniqueness of the solution, there is never a guarantee that the problem becomes convex and solvable in a reliable manner. In a non-convex solution process, local minima are often an issue. This becomes less manageable when the size of the problem increases, where the unique solution is not obtainable even if it exists. Moreover, in the presence of noise, local minima may degrade to false solutions. This means that they show the same magnitude-only reconstruction deviation as the true global solution. Even if the

solution is unique in the noiseless case, it is often not for real world measurement data.

In general, this leads to several unfavorable requirements for ‘pure’ phaseless field measurements. We need an extreme measurement accuracy, i.e., high signal-to-noise ratio (SNR) and a high positioning accuracy, which easily negates the advantage of a less sophisticated measurement hardware. Also, large oversampling ratios are necessary, which increase the measurement time (and cost). At least four-fold oversampling is a typical choice. Furthermore, the computational complexity of phase retrieval algorithms is a concern, since they often scale with the square of the number of unknowns or observations [29], [30] or even worse [31], [32]. This renders well thought out algorithms useless in practice. Overall, phaseless antenna measurements seem to be impractical if the task goes beyond checking a few magnitude samples for plausibility.

Fortunately, there are ways to tackle the issues of phase retrieval in antenna measurements. The lacking reliability of phase reconstructions could be considered as solved in case a convex algorithm with feasible computational effort and sampling requirements was employed. While convex algorithms, which work well with Gaussian and other purely magnitude-only data, do exist, e.g., PhaseLift [29], PhaseCut [30], PhaseMax [33], [34], they do not properly function with the data structures arising from electromagnetic field measurements. However, we have recently proposed a linearized and, thus, convex algorithm which works with the wide class of partially coherent data, i.e., containing subsets of the observation data which are linked coherently [17], [21], [24]. No overall phase reference is employed and synchronized oscillators at a probe array or clever combination circuits in connection with RF power detectors are sufficient to collect this kind of measurement data. The subset of the observations may for instance be collected with a probe antenna array.

As suggested in [21], mild conditions have to be fulfilled for our previously introduced linearized phase retrieval algorithm to work and to yield a unique solution. In the scope of electromagnetic field transformations, these conditions are actually not always fulfilled in practice. The hurdle is to sample the field such that the measurement subsets have a certain relationship. In this paper, we study the effect of the probe array design and derive simple guidelines with which all subsets are sufficiently independent and sufficiently connected at the same

time. Then, all relevant information about an antenna under test (AUT) can be collected. In Section II, we briefly revise the inverse source problem of antenna measurements in the complex and phaseless cases. Section III presents the linearized partially coherent phase retrieval algorithm. In Section IV, we discuss different probe antenna array designs as well as their respective advantages and drawbacks. Numerical results based on these considerations are presented in Section V.

II. SOURCE RECONSTRUCTION WITH AND WITHOUT PHASE

For a general complex inverse source problem, we choose a suitable equivalent source representation and retrieve the source coefficients collected in the vector $\mathbf{z} \in \mathbb{C}^{N \times 1}$ by solving the system of equations

$$\mathbf{A}\mathbf{z} = \mathbf{b}, \quad (1)$$

where $\mathbf{b} \in \mathbb{C}^{CM \times 1}$ are the observations and $\mathbf{A} \in \mathbb{C}^{CM \times N}$ is the respective discretized forward operator matrix. At each measurement location, the C measurement signals are recorded coherently.¹ Typically, the number of unknowns N and observations CM do not match, and both the source representation and the observations are often oversampled. To handle the non-trivial null-space in the matrix \mathbf{A} , an expedient way is to use the normal-error system of equations in combination with an iterative solver and fast methods. After retrieving the sources, quantities of interest such as the far-field pattern of an antenna under test (AUT) can be computed.

When external circumstances do not permit to measure the phase, we consider magnitude-only observations $|\mathbf{b}|$. Similarly as in the complex case, we aim to reconstruct the source coefficients \mathbf{z} . To do so, we have to solve the non-linear system of equations

$$|\mathbf{A}\mathbf{z}| = |\mathbf{b}|. \quad (2)$$

Often, the first step of the phaseless source reconstruction is to retrieve the phase of the observation vector. Then, subsequent steps can be carried out just as for the complex case.

Equation (2) is non-linear and non-convex for forward operators modeling electromagnetic radiation, even when strong oversampling and multiple measurement surfaces are employed. No matter which phase retrieval algorithm is employed—even the most computationally expensive ones, whose application is limited to electrically small scenarios—, the solution process struggles with the problem of local minima and/or false solutions, in particular in the presence of noise. Unlike for the complex case, where measurement errors for a given signal-to noise ratio lead to a fluctuation in the solution accuracy of several decibels, the phase retrieval process can always fail completely without any indication when we have only access to phaseless measurement data.

¹The case $C = 1$ for a single antenna and $C = 2$ for a dual-polarized probe are of course included here.

III. LINEARIZED PHASE RETRIEVAL WITH PARTIALLY COHERENT OBSERVATIONS

For our linearized phase retrieval algorithm, we have to introduce two new quantities. First, we need the projection matrix $\mathbf{P}_{\ker \mathbf{A}} = \mathbf{A}\mathbf{A}^+ - \mathbf{I}$ into the null space of \mathbf{A} , where \mathbf{A}^+ is a pseudo-inverse of \mathbf{A} . Second, the observations are now captured in M coherent subsets of each C samples. In the m th subset, the first sample is assigned an arbitrary phase, say zero, and a phase unknown $[\psi]_c$. These phaseless samples are collected in the vector $\mathbf{b}_1 \in \mathbb{R}^{M \times 1}$. All further samples are complex-valued with reference to the phase of the first sample. These are collected, for $2 \leq c \leq C$, in respective vectors $\mathbf{b}_c \in \mathbb{C}^{M \times 1}$. The vectors containing either real or complex measurement values are stored in the stacked-diagonal matrix $\mathbf{B} \in \mathbb{C}^{M \times CM}$,

$$\mathbf{B} = [\text{diag } \mathbf{b}_1 \quad \text{diag } \mathbf{b}_2 \quad \dots \quad \text{diag } \mathbf{b}_C]^T, \quad (3)$$

where $\text{diag}(\cdot)$ creates a diagonal matrix from a vector. The unique non-trivial solution to the homogeneous system of equations

$$\mathbf{P}_{\ker \mathbf{A}} \mathbf{B} \psi = \mathbf{0} \quad (4)$$

reconstructs the complex observation vector as $\mathbf{b} = \mathbf{B} \psi$ or $\mathbf{b} = \mathbf{B} \text{diag}(|\psi|)^{-1} \psi$ [21]. In practice, we search for a non-trivial vector in the kernel of $\mathbf{P}_{\ker \mathbf{A}} \mathbf{B}$ by setting one phase unknown to one, e.g., the one for the observation with the largest magnitude. Since the source model is hidden inside the projection matrix, and most of choices localized equivalent source models perform very similarly [35]–[37], it does not matter much which kind of equivalent sources are chosen for building the projection matrix.

For a unique solution to exist, the necessary but not sufficient condition

$$M(C - 1) \geq N - 1 \quad (5)$$

has to be fulfilled, where we can interpret N as the number of degrees of freedom arising from the chosen AUT source representation [21]. For the lowest case $C = 2$, we have to take at least $M \approx N$ subsets of measurements. For higher numbers of C , this number drops to a lower value, since more samples are taken in each subset, i.e., each position of the probe array.

IV. PROBE ARRAY DESIGN

The key of our linear phase retrieval approach with partially coherent information is to measure phase differences between sets of probes in order to reduce the ambiguities of the phase retrieval problem. Since our measurement problem is kind of four-dimensional (two position dimensions, if we assume a measurement surface, plus two polarization dimensions on this surface), phase differences should be measured in a way that all four dimensions of the measurement space are appropriately interconnected by known phase differences. As such, an appropriate antenna array for the measurement of the phase differences should feature a two-dimensional arrangement of probe antennas and feature polarization diversity in order to

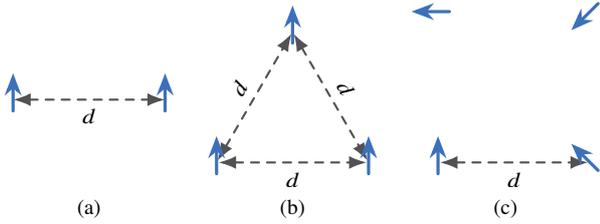


Fig. 1. Three of the investigated arrays of Hertzian dipole probe antennas. (a) Two-element array. (b) Three-element array. (c) Four-element array with varying polarization.

provide for the desirable interconnection of the measurement dimensions.

The utilization of linear arrays, as found in the literature [10]–[12], [15], [19], [20], [23], can not be considered sufficient in this respect. This is particularly important when the measurements are taken on regular sampling grids. Arrays capturing only the same polarization with all probe elements also lack a coherent interconnection for some parts of the data.

From these considerations, we conclude that a well-suited probe array must measure the two (typically tangential) polarizations on the measurement surface and must connect each polarization in the two spatial directions. This leads to arrays with at least $C = 4$ elements. However, even with $C = 4$ as well as spatial and polarization diversity, an inappropriate design of the probe array may lead to poor phase reconstruction results when the excitation interacts adversely with the field sampling [24].

Figure 1 gives an overview over the studied probe array configurations. All of them consist of Hertzian dipoles placed (approximately) tangentially on the measurement surface, in a distance of $d = 0.2\lambda$ to each other.

The two-element array and the three-element array are similar to probe arrays found in [21]. Since they capture only one polarization, they are also rotated by 90° at each measurement location for a second observation. The two-element array ($C = 2$) shown in Fig. 1(a) exhibits two drawbacks. The samples are connected only in one direction (in a partially coherent manner), and the coherent subsets only link samples with the same polarization information. The first drawback is mitigated in the three-element array ($C = 3$) depicted in Fig. 1(b). However, there is still no coherent connection between the two polarizations.

The four-element array ($C = 4$) from Fig. 1(c) contains Hertzian dipoles which are rotated by 45° for each further element. This array establishes coherent connections between samples in two directions and between both linear polarizations by clever linear combinations/rotations of each element. Hence, it is not necessary to rotate this probe array to measure the full polarization information. As an alternative, we randomly pick just one pair out of the four element array at each location randomly ($C = 2$, random). This array inherits some of the advantages of the well-designed four-element array but can be realized with a coherent two-channel receiver.

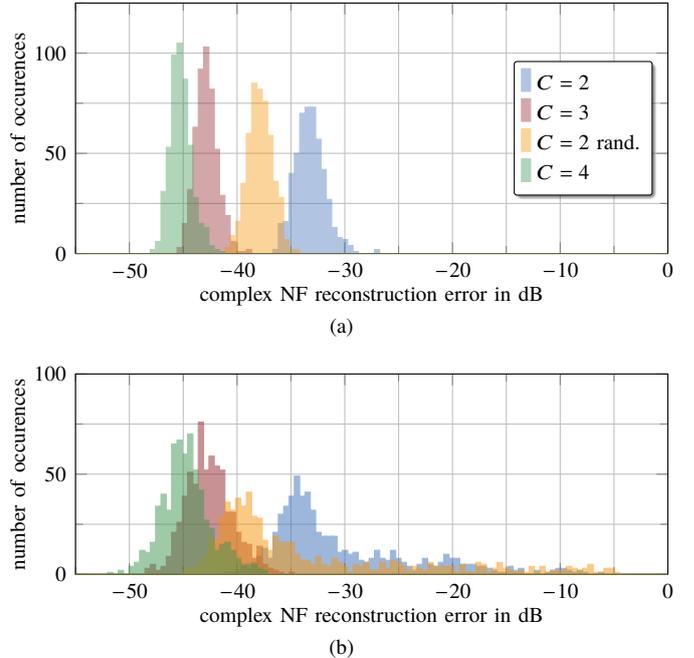


Fig. 2. Histograms of the reconstruction error for regular sampling. (a) 500 random excitations. (b) Individual excitations for all unknowns.

V. NUMERICAL RESULTS

For the following numerical study, we consider a spherical multipole expansion of order 17 as the AUT source model, which has $N = 646$ unknowns. The observations of the four probe arrays are taken on an observation sphere with a radius of 5λ . The oversampling ratio of $CM/N = 2.5$ is large enough that (5) is always fulfilled. Since C varies, the number of measurement locations M varies accordingly for the probe arrays. The synthetic measurement values are disturbed by white Gaussian noise with an SNR of 60 dB.

In Fig. 2, the histogram of complex NF reconstruction errors $20 \log_{10}(\mathbf{b}_{\text{rec}} - \mathbf{b}_{\text{true}})$ of random excitations is shown. The measurement locations are chosen on a regular grid. For the results of Fig. 2(a), all coefficients of the true solution are chosen randomly 500 times. The probe array with four elements clearly performs the best with a geometrically averaged reconstruction error of -45.2 dB. The versions $C = 3$, $C = 2$ (randomly picked), and $C = 2$ exhibit worse errors of -42.8 dB, -37.9 dB, and -33.2 dB. The histogram in Fig. 2(b) shows the NF errors of the phase reconstruction, when each of the 646 multipoles is excited individually. While the order of the probe arrays from best to worst is similar, the negative outliers in the right tail are more numerous for the arrays with $C = 2$, leading to much increased average errors of -30.2 dB ($C = 2$) and -33.5 dB ($C = 2$, random). In comparison, the errors for $C = 3$ and $C = 4$ are almost unchanged with -42.6 dB and -44.7 dB.

Since regular sampling is a rather demanding case for the partially coherent linkage of the measurement data, in particular when the rings of the grid are not connected at all for $C = 2$, the results of the same investigations but for Fibonacci

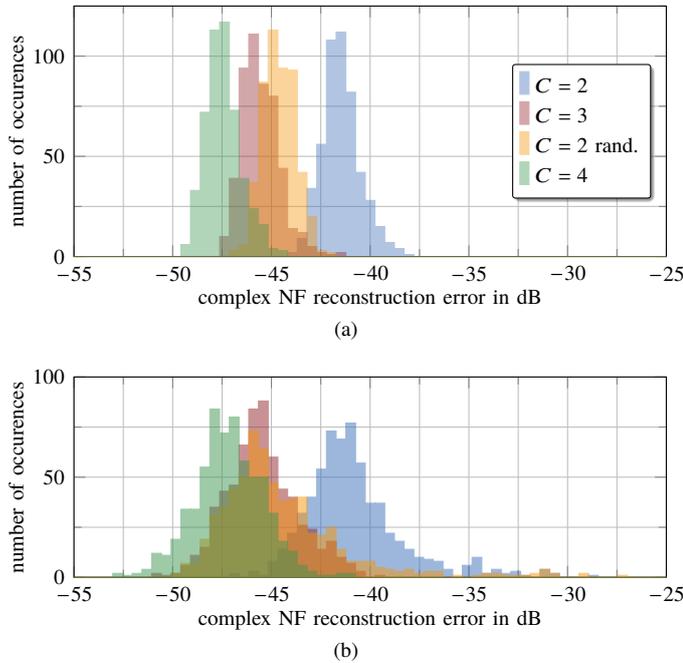


Fig. 3. Histograms of the reconstruction error for Fibonacci spiral sampling. (a) 500 random excitations. (b) Individual excitations for all unknowns.

spiral sampling are shown in Fig. 3. The phase reconstruction is throughout more accurate with this spiral sampling (and the same number of observations, also $CM/N = 2.5$). The approaches with $C = 2$ benefit the most. The $C = 2$ probe with random picks even performs almost as well as the $C = 3$ probe array. Most notably, the extreme outliers for the reconstruction of single multipole excitations are reduced by more than 20 dB.

VI. CONCLUSION

We have discussed criteria that are crucial for the design of probe antenna arrays employed in partially coherent phase retrieval and NF antenna measurements: The probe array has to be able to coherently connect fields in two independent directions and for both polarizations on the measurement surface. The theoretical discussion lead to four probe arrays with favorable (four elements) and less favorable (two elements) properties. Numerical phase retrieval results for synthetic measurement data have been used to validate the better performance of probe arrays which fulfill the mentioned design criteria.

ACKNOWLEDGMENT

The authors thank the German Federal Ministry for Economic Affairs and Climate Action for supporting this work in part under Grant 50RK1923 and Grant 50RK2221C.

REFERENCES

- [1] T. Fritzel, H. Steiner, J. Hartmann, and J. Habersack, "Revolutionary new outdoor testing with a mobile airborne nearfield test facility (ANTF)," in *Proc. 24th Ann. Symp. Antenna Meas. Techn. Assoc. (AMTA)*, Cleveland, OH, USA, Nov. 2002.
- [2] G. Virone, A. M. Lingua, M. Piras, A. Cina, F. Perini, J. Monari, F. Paonessa, O. A. Peverini, G. Addamo, and R. Tascone, "Antenna pattern verification system based on a micro unmanned aerial vehicle (UAV)," *IEEE Antennas Wirel. Propag. Lett.*, vol. 13, pp. 169–172, 2014.
- [3] A. Geise, O. Neitz, J. Migl, H. Steiner, T. Fritzel, C. Hunscher, and T. F. Eibert, "A crane-based portable antenna measurement system – system description and validation," *IEEE Trans. Antennas Propag.*, vol. 67, no. 5, pp. 3346–3357, May 2019.
- [4] G. Álvarez-Narciandi, J. Laviada, Y. Álvarez-López, and F. Las-Heras, "Portable freehand system for real-time antenna diagnosis and characterization," *IEEE Trans. Antennas Propag.*, vol. 68, no. 7, pp. 5636–5645, Jul. 2020.
- [5] F. T. Faul, H.-J. Steiner, and T. F. Eibert, "Near-field antenna measurements with manual collection of the measurement samples," *Adv. Radio Sci.*, vol. 18, pp. 17–22, 2020.
- [6] S. Punzet, F. T. Faul, T. Mittereder, C. Oettl, M. Ganser, M. Häusler, and T. F. Eibert, "Fully Coherent UAV-Based Near-Field Measurement and Transformation of the S67-15 m Ground Station Antenna at the German Space Operations Center in Weilheim," in *Proc. 16th Eur. Conf. Antennas Propag. (EuCAP)*, Madrid, ES, Mar. 2022.
- [7] R. A. M. Mauermayer and J. Kornprobst, "A cost-effective tethered-UAV-based coherent near-field antenna measurement system," *IEEE Open J. Antennas Propag.*, vol. 3, pp. 984–1002, 2022.
- [8] C. G. Parini, S. F. Gregson, and A. K. Brown, "Untethered near-field drone-based antenna measurement system for microwave frequencies using multiple reference antennas for phase and drone location recovery," *IET Microw. Antennas Propag.*, pp. 798–811, Oct. 2022.
- [9] T. Isernia, G. Leone, and R. Pierri, "Radiation pattern evaluation from near-field intensities on planes," *IEEE Trans. Antennas Propag.*, vol. 44, no. 5, pp. 701–710, May 1996.
- [10] S. Costanzo and G. Di Massa, "An integrated probe for phaseless plane-polar near-field measurements," *Microw. Opt. Technol. Lett.*, vol. 30, no. 5, pp. 293–295, Sep. 2001.
- [11] S. Costanzo, G. Di Massa, and M. D. Migliore, "A novel hybrid approach for far-field characterization from near-field amplitude-only measurements on arbitrary scanning surfaces," *IEEE Trans. Antennas Propag.*, vol. 53, no. 6, pp. 1866–1874, Jun. 2005.
- [12] A. Paulus, J. Knapp, and T. F. Eibert, "Phaseless near-field far-field transformation utilizing combinations of probe signals," *IEEE Trans. Antennas Propag.*, vol. 65, no. 10, pp. 5492–5502, Oct. 2017.
- [13] R. Moretta and R. Pierri, "Performance of phase retrieval via phaselift and quadratic inversion in circular scanning case," *IEEE Trans. Antennas Propag.*, vol. 67, no. 12, pp. 7528–7537, Dec. 2019.
- [14] A. Bangun, C. Culotta-López, A. Behboodi, R. Mathar, and D. Heberling, "On phaseless spherical near-field antenna measurements," in *Proc. 13th Eur. Conf. Antennas Propag. (EuCAP)*, Krakow, PL, Apr. 2019.
- [15] R. Tena-Sánchez, M. Sierra-Castañer, and L. J. Foged, "Relative phase reconstruction based on multiprobe solutions and post-processing techniques," in *Proc. 14th Eur. Conf. Antennas Propag. (EuCAP)*, Copenhagen, DK, Mar. 2020.
- [16] B. Fuchs, M. Mattes, S. Rondineau, and L. Le Coq, "Phaseless near-field antenna measurements from two surface scans — numerical and experimental investigations," *IEEE Trans. Antennas Propag.*, vol. 68, no. 3, pp. 2315–2322, Mar. 2020.
- [17] J. Knapp, A. Paulus, J. Kornprobst, U. Siart, and T. F. Eibert, "Multi-frequency phase retrieval for antenna measurements," *IEEE Trans. Antennas Propag.*, vol. 69, pp. 488–501, Jan. 2021.
- [18] R. Pierri, G. Leone, and R. Moretta, "Phaseless near-field techniques from a random starting point," in *Proc. XXXIIIrd General Assembly Sci. Symp. Int. Union Radio Sci. (URSI GASS)*, Rome, IT, Sep. 2020.
- [19] R. Tena-Sánchez, F. Rodríguez Varela, L. J. Foged, and M. Sierra Castañer, "Reconstruction of relative phase of self-transmitting devices by using multiprobe solutions and non-convex optimization," *Sensors*, vol. 21, no. 7, p. 2459, 2021.
- [20] F. Rodríguez Varela, M. J. López Morales, R. Tena-Sánchez, A. T. Muriel Barrado, E. de la Fuente González, G. Posada Quijano, C. Zarzuelo Torres, M. Sierra Pérez, and M. Sierra Castañer, "Multiprobe measurement system based on single-cut transformation for fast testing of linear arrays," *Sensors*, vol. 21, no. 5, p. 1744, 2021.
- [21] J. Kornprobst, A. Paulus, J. Knapp, and T. F. Eibert, "Phase retrieval for partially coherent observations," *IEEE Trans. Signal Proc.*, vol. 69, pp. 1394–1406, 2021.
- [22] L. Ciorba, G. Virone, F. Paonessa, M. Righero, E. de Lera Acedo, S. Matteoli, E. C. Beltran, P. Bolli, G. Giordanengo, G. Vecchi, A. Ma-

- gro, R. Chiello, O. A. Peverini, and G. Addamo, "Large horizontal near-field scanner based on a non-tethered unmanned aerial vehicle," *IEEE Open J. Antennas Propag.*, vol. 3, pp. 568–582, May 2022.
- [23] F. R. Varela, B. Galocha-Iragüen, and M. Sierra-Castañer, "Single-cut phaseless near-field measurements using specialized probes," in *Proc. 44th Ann. Meeting Symp. Antenna Meas. Techn. Assoc. (AMTA)*, Denver, CO, USA, Oct. 2022.
- [24] A. Paulus, J. Kornprobst, and T. F. Eibert, "Electromagnetic field transformations of near-field data without global reference for magnitude and phase," in *Proc. 44th Ann. Meeting Symp. Antenna Meas. Techn. Assoc. (AMTA)*, Denver, CO, US, Oct. 2022.
- [25] S. Costanzo, G. Lopez, and G. Di Massa, "Spatial domain indirect holography for a full phaseless approach to biomedical microwave imaging," *IEEE Open J. Antennas Propag.*, vol. 3, pp. 604–612, May 2022.
- [26] F. Bevilacqua, A. Capozzoli, C. Curcio, F. D'Agostino, F. Ferrara, R. Guerriero, A. Lisenò, M. Migliozzi, and Y. Vardaxoglou, "Reflectivity reconstruction from only amplitude non-redundant near-field data: numerical validation," in *Proc. 44th Ann. Meeting Symp. Antenna Meas. Techn. Assoc. (AMTA)*, Denver, CO, USA, Oct. 2022.
- [27] E. J. Candès, X. Li, and M. Soltanolkotabi, "Phase retrieval via Wirtinger flow: Theory and algorithms," *IEEE Trans. Inf. Theory*, vol. 61, no. 4, pp. 1985–2007, Apr. 2015.
- [28] G. Wang, G. B. Giannakis, Y. Saad, and J. Chen, "Phase retrieval via reweighted amplitude flow," *IEEE Trans. Signal Proc.*, vol. 66, no. 11, pp. 2818–2833, Jun. 2018.
- [29] E. J. Candes, T. Strohmer, and V. Voroninski, "Phaselift: Exact and stable signal recovery from magnitude measurements via convex programming," *Commun. Pure Appl. Math.*, vol. 66, no. 8, pp. 1241–1274, Aug. 2013.
- [30] I. Waldspurger, A. d'Aspremont, and S. Mallat, "Phase recovery, maxcut and complex semidefinite programming," *Mathematical Programming*, vol. 149, no. 1, pp. 47–81, 2015.
- [31] J. Knapp, A. Paulus, and T. F. Eibert, "Reconstruction of squared field magnitudes and relative phases from magnitude-only near-field measurements," *IEEE Trans. Antennas Propag.*, vol. 67, no. 5, pp. 3397–3409, May 2019.
- [32] R. Palmeri, G. Battaglia, A. F. Morabito, and T. Isernia, "Reflector antennas characterization and diagnostics using a single set of far-field phaseless data and crosswords-like processing," *IEEE Trans. Antennas Propag.*, vol. 70, pp. 8424–8439, Sep. 2022.
- [33] T. Goldstein and C. Studer, "Phasemax: Convex phase retrieval via basis pursuit," *IEEE Trans. Inf. Theory*, vol. 64, no. 4, pp. 2675–2689, Apr. 2018.
- [34] S. Bahmani and J. Romberg, "Phase retrieval meets statistical learning theory: A flexible convex relaxation," in *Proc. 20th Int. Conf. Artificial Intell. Stat. (AISTATS)*, vol. 54, Fort Lauderdale, FL, USA, Apr. 2017, pp. 252–260.
- [35] J. L. Araque Quijano and G. Vecchi, "Field and source equivalence in source reconstruction on 3D surfaces," *Prog. Electromagn. Res.*, vol. 103, pp. 67–100, 2010.
- [36] J. Kornprobst, R. A. M. Mauermayer, O. Neitz, J. Knapp, and T. F. Eibert, "On the solution of inverse equivalent surface-source problems," *Prog. Electromagn. Res.*, vol. 195, pp. 47–65, 2019.
- [37] J. Kornprobst, J. Knapp, R. A. M. Mauermayer, O. Neitz, A. Paulus, and T. F. Eibert, "Accuracy and conditioning of surface-source based near-field to far-field transformations," *IEEE Trans. Antennas Propag.*, vol. 69, no. 8, pp. 4894–4908, Aug. 2021.